Parametric Transforms

Definition of a parametric transform

A transform, $t(a,x)$ is a parametric transform if:

$$t(0,x) = x;$$
$$t(a, t(b,x)) = t(a+b, x)$$

The inverse transform is given by

$$t_{inv}(a,x) = t(a_{inv},x)$$

Consequently

$$tinv(a,tinv(a,x)) = x$$
$$t(ainv,tinv(a,x)) = t(ainv+a,x) = t(0,x)$$
$$ainv = -a, \text{ and}$$
$$t_{inv}(a,x) = t(-a,x)$$

For example, consider the transform, $g(a,x) = x\char`^(k\char`^a)$, where k is a scalar and ^ is exponentation. It follows that $g(a,x)$ is a parametric transform, since

$$g(0,x) = x\char`^(k\char`^0) = x\char`^(1) = x$$
$$g(a,g(b,x)) = (x\char`^(k\char`^b))\char`^(k\char`^a) = x\char`^(k\char`^(a+b)) = g(a+b,x)$$

Distortion

If $t(a,x)$ is a polynomial of degree n, then $t(a, t(b,x))$, is of degree 2n. Therefore $t(a+b, x)$ is not well defined. However, suppose that higher order polynomial terms are unwanted distortions. That is, we seek a solution

$$t(a, t(b,x)) = t(a+b, x) + d(a,b,x), \text{ where d is sufficiently small. or equivalently,}$$
$$t(a, t(b,x)) \approx t(a+b, x)$$

The sense of this approximation is that the composite transform is a "best approximation" without generating higher order distortions.

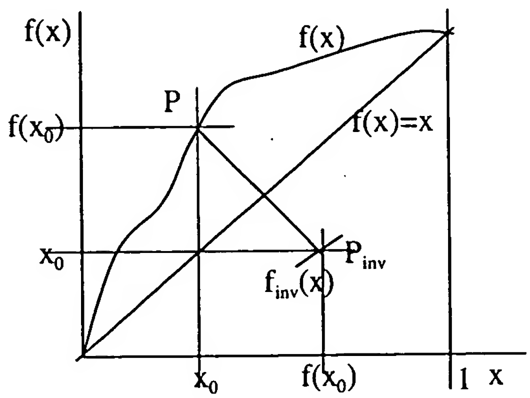
However, we want to keep the exact solution $t_{inv}(a,x) = t(-a,x)$. So the requirements for a parametric transform are

$$t(0,x) = x;$$
$$t_{inv}(a,x) = t(-a,x)$$
$$t(a, t(b,x)) \approx t(a+b, x), \text{ a sufficiently good approximation}$$

General derivation of parametric transforms

It will now be shown that a parametric transform can be calculated from any transform whose inverse exists. For convenience the function f(x) has values f(0)=0, and f(1)=1.

The following figure shows such a function defined over interval (0,1)



Choose a point, $P = (x_0,f(x_0))$, on the curve f(x). Then the point, $P_{inv} = (f(x_0),x_0)$ is on the curve $f_{inv}(x)$. A straight line can be drawn through the points, P and P_{inv}. This line can be written parametrically as

$$((1+a)x_0 + (1-a)f(x_0), (1-a)x_0 + (1+a)f(x_0))/2$$

Note that

a=0 gives the point $(x_0+f(x_0), x_0+f(x_0))/2$ which lies on the line f(x) = x
a=1 gives the point $(x_0,f(x_0)) = P$
a=-1 gives the point $(f(x_0),x_0) = P_{inv}$

We now generalize this result to all points. Since PP_{inv} is perpendicular to the line f(x) = x then a 45o axis rotation reduces the parametric variation to a single dimension. Namely

$$u = (x+f(x))/\sqrt{2}$$
$$g(u) = (f(x)-x)/\sqrt{2}$$

Substituting and rearranging terms gives the convenient form

$$f(x) = u\sqrt{2} - x$$
$$g(u) = u - x\sqrt{2}$$

By substitution

$$g((f(x)+x)/\sqrt{2}) = (f(x)-x)/\sqrt{2}, \text{ and}$$
$$f((u-g(u)/\sqrt{2}) = (u+g(u))/\sqrt{2}$$

The first of these equations can generate the function g from (x,f(x)) pairs, and the second equation can generate the function f from (u,g(u)) pairs.

We introduce a parameter $a = (-1,1)$ as follows:

$$ag(u) = u - x\sqrt{2}$$
$$t(a,x) = u\sqrt{2} - x$$

The corresponding generator function is

$$t(a, (u-ag(u)/\sqrt{2}) = (u+ag(u)/\sqrt{2}$$

We now show that t(a,x) is a parametric transform. Firstly

$$u= x\sqrt{2}, \text{ for } a= 0, \text{ thus}$$
$$t(0,x) = u\sqrt{2} - x = (x\sqrt{2})\sqrt{2} - x = x$$

Secondly

$$t(-a, t(a, (u-ag(u)/\sqrt{2}) = t(-a,(u+ag(u)/\sqrt{2}) = (u-ag(u)/\sqrt{2}$$

thus, by inspection,

$t(-a,t(a,x)) = x$, so $t(-a,x)$ is $t_{inv}(a,x)$

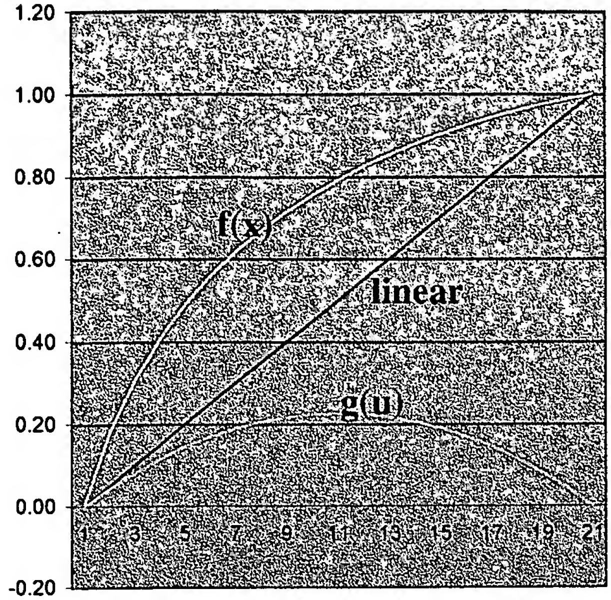
Example derivation

For this example we assert the value of g(u) and derive the corresponding f(x)

$$g(u) = au(1-u/\sqrt{2})/\sqrt{2}$$

$$f(x) = (a(1-x) -\sqrt{2} + \sqrt{(2 + a2 - a2\sqrt{2} + 4ax\sqrt{2})})/a$$

These equations , along with f(x)=x, are shown in the following graphs for a=1.


1.20
1.00
0.80
0.60
0.40
0.20
0.00
-0.20
f(x)
linear
g(u)
1
3
5
7
9
11
13
15
17
19
21